Assuming that no transmission zero coincides with model eigenvalues, define matrices Ω and S_{ij} for the open-loop system (A,B,C) as in Ref. 2. Then, the feedforward control with an inverse of the closed-loop system, $U_f(s) = [Y_c/U_f(s)]^{-1}[Y_m/U_m(s)]\ U_m(s)$, is shown to be generated by the same equation as in the Comment:

$$U_f(s) = (S_{21}+LS_{11})X_m(s) + (S_{22}+LS_{12})U_m(s)$$

$$+(\Omega_{21}+L\Omega_{11})\nu(s)$$

$$\nu(s) \triangleq (I-s\Omega_{11})^{-1}S_{12}\dot{U}_m(s)$$

This shows how an arbitrary feedback gain L reflects in the feedforward gains. A little manipulation using relations among Ω, S_{ij}, and (A,B,C) yields the following quite interesting expression for the above $U_f(s)$:

$$U_f(s) = [B^{*-1}(A_m^* + KM_m) + (L-K_x)S_{11}]X_m(s)$$

$$+[B^{*-1}B_m^* + (L-K_x)S_{12}]U_m(s) + (L-K_x)\Omega_{11}\nu(s)$$

where

$$K_x \triangleq B^{*-1}(A^* + KM)$$

Matrices A^*, B^*, M, K, K_x and $(\cdot)_m$ are all defined in Ref. 1.

The above expression suggests that if $L=K_x$, there is no need to have the time derivatives of u_m, and the control law becomes the same as that offered by Ref. 1. When the plant has no transmission zero, there always exists a unique matrix K to satisfy $L=K_x$ for an arbitrary feedback gain matrix L since M is nonsingular for an observable system. The matrix K is not necessarily of block-diagonal form as is noted in Ref. 1. On the other hand, when there is any (stable) transmission zero the matrix M is, in general, singular. The feedback gain matrix L to eliminate the time derivatives of u_m or ν is *not completely arbitrary* since the matrix K to satisfy $L=K_x$ may not exist. Thus, it has been made clear that the feedback gains offered in Ref. 1 are quite significant to avoid $\dot{u}_m$ in the model following control.

Since, in practice, the time derivatives of u_m are often not available, the system designer may wish to have a feedback control with the gain $L=K_x$. However, some of the transmission zeros may be close to the origin and the imaginary axis, so that the transient output error response may persist and the closed-loop response to system noise (e.g., aircraft gust response) be considerably dominant. If that is the case, one may opt for a feedforward realization with a tight feedback loop ($L \neq K_x$) at the expense of generating the time derivatives of u_m. Thus, in practical situations with stable transmission zeros, a compromise between the feedforward[2] and/or the feedback[1] realizations of the model following control may be called for.

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Comment on "Time-Optimal Orbit Transfer Trajectory for Solar Sail Spacecraft"

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Introduction

In Ref. 1, Jayaraman discusses minimum-time heliocentric transfers between the Earth's orbit and the orbit of Mars using a solar sail propulsion system. The orbits of these planets are assumed to be circular and coplanar. Planetary masses are neglected. Jayaraman applies the calculus of variations to search for minimum-time trajectories for two values of solar sail characteristic thrust acceleration.[2] Jayaraman's solutions differ from those obtained by Kelley[3,4] and Zhukov and Lebedev.[5] His transfer times are about 10% larger and his sail orientation histories significantly different. Jayaraman asserts that he has found the minimum-time solutions and that the earlier, shorter transfer times in Refs. 3-5 were probably for trajectories which do not accurately satisfy the required boundary conditions. We show here that the solutions in Refs. 3-5 (and also in Ref. 2) are correct and that a transversality condition of variational calculus has been applied incorrectly in Ref. 1.

Reference 1 also discusses the merits of solar sailing for carrying out interplanetary space missions, relative to electric propulsion, in particular. We argue here that the orbit transfer problems considered contain too many simplifications to allow a realistic comparison of these advanced propulsion technologies and note that a number of more sophisticated studies of solar sailing missions were carried out in the late 1970's.

Conditions of Optimality

The normalized equations of motion for a planar heliocentric transfer by means of a solar sail spacecraft are

$$\mathrm{d}x_1/\mathrm{d}t = x_2 \tag{1}$$

$$\mathrm{d}x_2/\mathrm{d}t = x_3^2/x_1 - 1/x_1^2 + \beta\cos^3 u/x_1^2 \tag{2}$$

$$\mathrm{d}x_3/\mathrm{d}t = -x_2x_3/x_1 + \beta\cos^2 u\sin u/x_1^2 \tag{3}$$

where t, x_1, x_2, x_3, and u denote time, radial distance of the spacecraft from the sun, radial velocity, circumferential velocity, and angle between the outwardly directed normal to

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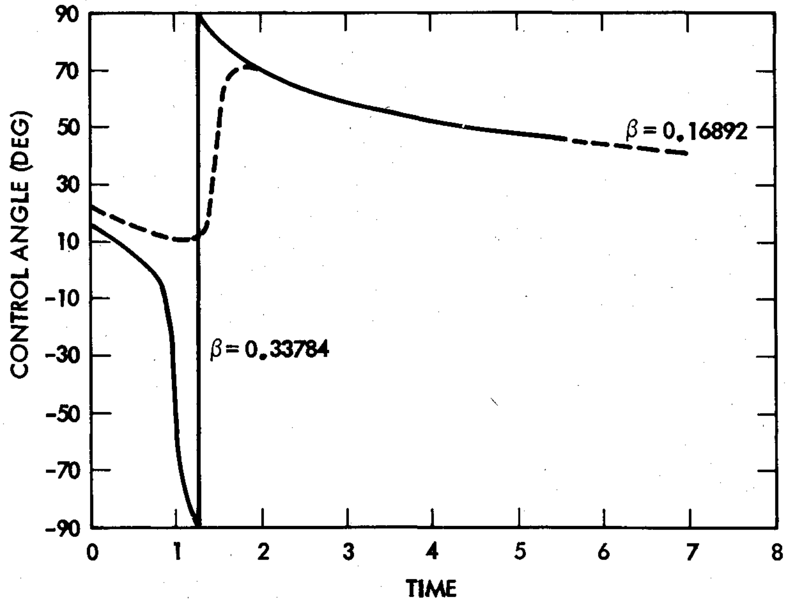

Fig. 1 Control angle history.

the sail and the radius vector (increasing in a counterclockwise sense), respectively. (See Fig. 1 in Ref. 1.) β is the normalized characteristic acceleration of the sail. Equations (2) and (3) are valid for $|u| \leq \pi/2$. Equations (1-3) agree with their counterparts in Refs. 1 and 3-5, apart from a few inconsistencies regarding the angle u,[1,3,4] which are easily resolved.

Boundary conditions used in Ref. 1 for the orbits of Earth and Mars are

$$x_1(0)=1 \qquad x_2(0)=0 \qquad x_3(0)=1 \tag{4}$$

and (to four significant figures)

$$x_1(t_f)=1.525 \qquad x_2(t_f)=0 \qquad x_3(t_f)=0.8098 \tag{5}$$

The quantity to be minimized is

$$J=t_f \tag{6}$$

yielding the variational Hamiltonian

$$H=\lambda_1 x_2+\lambda_2[x_3^2/x_1-1/x_1^2+\beta\cos^3 u/x_1^2]$$
$$+\lambda_3[-x_2x_3/x_1+\beta\cos^2 u\sin u/x_1^2] \tag{7}$$

The optimal control is determined by minimizing H with respect to u.[6] Checking first for stationary values of H, we find that

$$\partial H/\partial u=(\beta/x_1^2)[\cos^2 u(-\lambda_2\sin u+\lambda_3\cos u)$$
$$-2\cos u\sin u(\lambda_2\cos u+\lambda_3\sin u)] \tag{8}$$

For $\beta>0$ and λ_2 and λ_3 not both zero, this quantity vanishes for

$$\cos u=0 \tag{9}$$

or

$$2\lambda_3\tan^2 u+3\lambda_2\tan u-\lambda_3=0 \tag{10}$$

H is thus stationary with respect to u when Eq. (9) holds, or when

$$\tan u=[-3\lambda_2\pm(9\lambda_2^2+8\lambda_3^2)^{1/2}]/4\lambda_3 \qquad \lambda_3\neq 0$$
$$=0 \qquad \lambda_3=0 \tag{11}$$

Among these various solutions, the value of u which minimizes H, subject to $|u|\leq\pi/2$, is

$$u=\tan^{-1}\{[-3\lambda_2-(9\lambda_2^2+8\lambda_3^2)^{1/2}]/4\lambda_3\} \qquad \lambda_3\neq 0 \tag{12}$$

$$=0 \qquad \lambda_3=0, \lambda_2<0 \tag{13}$$

$$=\pm\pi/2 \qquad \lambda_3=0, \lambda_2>0 \tag{14}$$

There is a sign difference between Eq. (12) and its counterpart in Ref. 5, due to the fact that the Hamiltonian is being maximized in Ref. 5. This multiplicity of stationary values of H reveals a difficulty in applying to this problem optimization algorithms which drive $\partial H/\partial u$ to zero but do not check for minimization of H—the wrong stationary solution may be found. It is conceivable that this happened in Ref. 1.

Differential equations for the Lagrange multipliers λ_1, λ_2, and λ_3 are given by[6]

$$d\lambda_i/dt=-\partial H/\partial x_i \qquad (i=1,2,3) \tag{15}$$

The terminal values of λ_1, λ_2, and λ_3 are unknown, since the terminal values of x_1, x_2, and x_3 are specified.[6] Since the final time is unspecified (and is to be optimized), the following transversality condition is applicable[6]:

$$H(t_f)+1=0 \tag{16}$$

Jayaraman has omitted the "1" in Eq. (16) (cf. Eqs. (14), (16), (18), and (21) in Ref. 1). Since the Hamiltonian function and the differential equations for λ_1, λ_2, and λ_3 are linear and homogeneous in λ_1, λ_2, and λ_3, Eq. (16) may be replaced by

$$H(t_f)+a=0 \tag{17}$$

for any $a>0$, without affecting the control history determined according to Eq. (12). The Lagrange multipliers are simply scaled by the factor a (they lose their geometrical significance as partial derivatives of the optimal return function[6] with respect to the state, for $a\neq 1$, however). Zhukov and Lebedev[5] choose a to make $\lambda_3(0)=-1$, for convenience. Sauer allows a to remain arbitrary (though positive). Unlike the approaches in Refs. 2 and 5, however, setting a to zero, as was in effect done in Ref. 1, will no longer yield the same control history.

Table 1 Minimum transfer times and Lagrange multipliers

Normalized characteristic acceleration	Transfer time	Initial Lagrange multipliers			Terminal Lagrange multipliers		
		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.16892	7.02232 (408 days)	−7.40981	−4.22855	−7.91115	−21.5481	8.50251	−46.6174
0.33784	5.57911 (324 days)	−3.68044	−2.59597	−2.56421	−14.9271	10.1662	−29.6344

Minimum-Time Trajectories

Jayaraman solved the minimum-time Earth-Mars solar sail orbit transfer problem for two values of normalized sail characteristic acceleration, $\beta = 0.16892$ and $\beta = 0.33784$, which correspond to characteristic accelerations of about 1 mm/s^2 and 2 mm/s^2, respectively. Kelley[3,4] had previously determined a control history and a transfer time (about 410 days) for the 1 mm/s^2 case. Zhukov and Lebedev[5] had determined a control history and a transfer time (322 days) for the 2 mm/s^2 case. Reference 5 also includes a transfer time for the 1 mm/s^2 case (405 days) and several other cases, but no corresponding control histories are given. Sauer[2] has presented similar transfer times for these two cases. Jayaraman's transfer times are about 10% larger, in each case (445.5 and 355.4 days for characteristic accelerations of 1 and 2 mm/s^2, respectively). Thus, his solutions cannot be optimal, unless the solutions in Refs. 2-5 are incorrect.

We have recently resolved both problems. We used a neighboring extremal algorithm based upon numerical differentiation,[6] in conjunction with Krogh's variable order, variable step size integrator, to obtain solutions on the UNIVAC 1100/81 computer at the Jet Propulsion Laboratory. We then confirmed these solutions with a nonlinear search algorithm based upon a shooting method, in conjunction with an eighth-order Adams-Moulton variable step size integrator, on the CDC 176 computer at the Aerospace Corporation. All computations were performed using double-precision arithmetic. Transfer times and initial and final Lagrange multipliers are given in Table 1 for the two cases. Control histories are given in Fig. 1. Our transfer times match those tabulated in Ref. 5 and plotted in Refs. 2-4 to within about 1%. Our control histories match those plotted in Ref. 4 (Eulerian solution) and Ref. 5 to within the error inherent in plotting and reading graphs, but differ substantially from those in Ref. 1. (The sail angle increases in a clockwise sense in Refs. 1, 3, and 4, but in a counterclockwise sense here and in Ref. 5.) Jayaraman observes that no statements are made in Refs. 3-5 regarding the accuracies to which the solutions are converged and suggests that the shorter transfer times therein are due to a failure to satisfy end-point constraints (4) and (5). Our endpoint constraints are satisfied to better than 10^{-9}, a level comparable to that in Ref. 1.

Our trajectories are stationary and satisfy the Pontryagin Minimum Principle. We have not applied a conjugate point check,[6,7] as was done in Ref. 8 for a related electric propulsion orbit transfer, which would guarantee that the solutions are at least locally minimizing. There is no apparent reason to believe that this is not the case, though. Since low thrust orbit transfer problems are known to have multiple stationary solutions in some circumstances, we did a limited amount of searching for other stationary solutions consistent with the Minimum Principle. We found none. It seems unlikely that the solutions in Ref. 1 are nonminimizing, but yet stationary, in view of the incorrect use of the transversality condition noted above.

Solar Sail Mission Design Studies

Certain conclusions are drawn in Ref. 1 regarding the relative merits of solar sailing and electric propulsion in interplanetary space missions, based upon the numerical results presented in Refs. 1 and 9. The electric propulsion trajectories in Ref. 9 are nonoptimal, it turns out,[10,11] although the Earth-Mars transfer time is only too large by about 2%. Included in Ref. 11 is a discussion of the difficulties which may be encountered in applying conjugate gradient methods to dynamic optimization problems with three or more terminal state constraints and a free final time. The state equations in Ref. 9 are for a nuclear-electric (as opposed to solar-electric) vehicle, since the thrust is assumed independent of heliocentric distance.

These three state variable problems have sufficient physical content to be useful in determining the general features of optimal low thrust orbit transfers and have been useful in more than a dozen comparisons of numerical optimization techniques,[10,11] but contain too many simplifications to be of use in designing realistic space missions or in assessing the relative merits of two advanced propulsion technologies. In particular, planetary orbits are neither circular nor coplanar, and planetary masses are nonzero. The latter consideration requires a chemical propulsion system for launch (and perhaps another one for arrival at the target body), the properties of which should be taken into account in an overall mission design optimization.[2] Moreover, minimization of flight time is likely to be of less consequence than maximization of delivered payload and minimization of overall mission cost and risk, especially for flights to the inner planets, which are relatively short regardless of propulsion mode.

A number of studies of solar sailing missions to the inner planets, Jupiter, Saturn, and comets and asteroids were carried out in the late 1970's, with emphasis focusing particularly on a rendezvous with Comet Halley in 1986. Sauer[2,12] presented detailed trajectory designs, based upon generalizations of the variational approach in Ref. 5, for such missions. The navigation of the Halley Rendezvous mission was discussed in Ref. 13. A number of detailed mission and flight system designs were presented in Refs. 14 and 15.

Acknowledgments

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Errata

Airplane Performance Sensitivities to Lateral and Vertical Profiles

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THE engines attributed to the United Airlines' 727 in this article were erroneously referred to as JT9D-7's. The engine actually installed is the JT8D-7. The authors regret any confusion this may have caused their readers.

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